Optimal power dispatch in networks of high-dimensional models of synchronous machines

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Tjerk Stegink and Claudio De Persis and Arjan van der Schaft

Abstract— **This paper investigates the problem of optimal frequency regulation of multi-machine power networks where each synchronous machine is described by a sixth order model. By analyzing the physical energy stored in the network and the generators, a port-Hamiltonian representation of the multi-machine system is obtained. Moreover, it is shown that the open-loop system is passive with respect to its steady states which allows the construction of passive controllers to control the multi-machine network. As a special case, a distributed consensus based controller is designed that regulates the frequency and minimizes a global quadratic generation cost in the presence of a constant unknown demand. In addition, the proposed controller allows freedom in choosing any desired connected undirected weighted communication graph.**

I. INTRODUCTION

The control of power networks has become increasingly challenging over the last decades. As renewable energy sources penetrate the grid, the conventional power plants have more difficulty in keeping the frequency around the nominal value, e.g. 50 Hz, leading to an increased chance of a network failure or even a blackout.

The current developments require a better understanding of more advanced models for the power network as the grid is operating more often near its capacity constraints. Considering high-order models of, for example, synchronous machines, that better approximate the reality allows us to establish results on the control and stability of power networks that are more reliable and accurate.

At the same time, incorporating economic considerations in the power grid has become more difficult. As the scale of the grid expands, computing the optimal power production allocation in a centralized manner as conventionally is done is computationally expensive, making distributed control far more desirable compared to centralized control. In addition, often exact knowledge of the power demand is required for computing the optimal power dispatch, which is unrealistic in practical applications. As a result, there is an increased desire for distributed real-time controllers which are able to compensate for the uncertainty of the demand.

In this paper, we propose an energy-based approach for the modeling, analysis and control of the power grid, both for the physical network as well as for the distributed controller design. Since energy is the main quantity of interest, the port-Hamiltonian framework is a natural approach to deal with the problem. Moreover, the port-Hamiltonian framework lends itself to deal with complex large-scale nonlinear systems like power networks [5], [13], [14].

The emphasis in the present paper lies on the modeling and control of (networked) synchronous machines as they play an important role in the power network since they are the most flexible and have to compensate for the increased fluctuation of power supply and demand. However, the full-order model of the synchronous machine as derived in many power engineering books like [2], [6], [8] is difficult to analyze, see e.g. [5] for a port-Hamiltonian approach, especially when considering multi-machine networks [4], [9]. Moreover, it is not necessary to consider the full-order model when studying electromechanical dynamics [8].

On the other hand of the spectrum, many of the recent optimal controllers in power grids that deal with optimal power dispatch problems rely on the second-order (non)linear swing equations as the model for the power network [7], [11], [12], [18], [19], or the third-order model as e.g. in [16]. However, the swing equations are inaccurate and only valid on a limited time interval up to the order of a few seconds so that asymptotic stability results are often invalid for the actual system [2], [6], [8].

Hence, it is appropriate to make simplifying assumptions for the full-order model and to focus on multi-machine models with intermediate complexity which provide a more accurate description of the network compared to the second- and third-order models [2], [6], [8]. However, for the resulting intermediate-order multi-machine models the stability analysis is often carried out for the linearized system, see [1], [6], [8]. Consequently, the stability results are only valid around a specific operating point.

Our approach is different as the nonlinear nature of the power network is preserved. More specifically, in this paper we consider a nonlinear sixth-order reduced model of the synchronous machine that enables a quite accurate description of the power network while allowing us to perform a rigorous analysis.

In particular, we show that the port-Hamiltonian framework is very convenient when representing the dynamics of the multi-machine network and for the stability analysis. Based on the physical energy stored in the generators and the transmission lines, a port-Hamiltonian representation of the multi-machine power network can be derived. More specifically, while the system dynamics is complex, the in-

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terconnection and damping structure of the port-Hamiltonian system is sparse and, importantly, state-independent.

The latter property implies shifted passivity of the system [17] which respect to its steady states which allows the usage of passive controllers that steer the system to a desired steady state. As a specific case, we design a distributed real-time controller that regulates the frequency and minimizes the global generation cost without requiring any information about the unknown demand. In addition, the proposed controller design allows us to choose any desired undirected weighted communication graph as long as the underlying topology is connected.

The main contribution of this paper is to combine distributed optimal frequency controllers with a high-order nonlinear model of the power network, which is much more accurate compared to the existing literature, and to prove asymptotic stability to the set of optimal points by using Lyapunov function based techniques.

The rest of the paper is organized as follows. In Section II the preliminaries are stated and a sixth order model of a single synchronous machine is given. Next, the multi-machine model is derived in Section III. Then the energy functions of the system are derived in Section IV, which are used to represent the multi-machine system in port-Hamiltonian form, see Section V. In Section VI the design of the distributed controller is given and asymptotic stability to the set of optimal points is proven. Finally, the conclusions and the possibilities for future research are discussed in Section VII.

II. PRELIMINARIES

Consider a power grid consisting of n buses. The network is represented by a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the set of nodes, $\mathcal{V} = \{1, ..., n\}$, is the set of buses and the set of edges, $\mathcal{E} = \{1, ..., m\} \subset \mathcal{V} \times \mathcal{V}$, is the set of transmission lines connecting the buses. The ends of edge $l \in \mathcal{E}$ are arbitrary labeled with a '+' and a '-', so that the incidence matrix D of the network is given by

$$D_{il} = \begin{cases} +1 & \text{if } i \text{ is the positive end of } l \\ -1 & \text{if } i \text{ is the negative end of } l \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Each bus represents a synchronous machine and is assumed to have controllable mechanical power injection and a constant *unknown* power load. The dynamics of each synchronous machine $i \in \mathcal{V}$ is assumed to be given by [8]

$$\begin{aligned} M_i\dot{\omega}_i &= P_{mi} - P_{di} - V_{di}I_{di} - V_{qi}I_{qi} \\ \dot{\delta}_i &= \omega_i \\ T'_{di}\dot{E}'_{qi} &= E_{fi} - E'_{qi} + (X_{di} - X'_{di})I_{di} \\ T'_{qi}\dot{E}'_{di} &= -E'_{di} - (X_{qi} - X'_{qi})I_{qi} \\ T''_{di}\dot{E}''_{qi} &= E'_{qi} - E''_{qi} + (X'_{di} - X''_{di})I_{di} \\ T''_{qi}\dot{E}''_{di} &= E'_{di} - E''_{di} - (X'_{qi} - X''_{qi})I_{qi}, \end{aligned} \tag{2}$$

see also Table I.

δ_i	rotor angle w.r.t. synchronous reference frame
ω_i	frequency deviation
P_{mi}	mechanical power injection
P_{di}	power demand
M_i	moment of inertia
X_{qi}, X_{di}	synchronous reactances
X'_{qi}, X'_{di}	transient reactances
X''_{di}, X''_{qi}	subtransient reactances
E_{fi}	exciter emf/voltage
E'_{qi}, E'_{di}	internal bus transient emfs/voltages
E''_{qi}, E''_{di}	internal bus subtransient emfs/voltages
V_{qi}, E_{di}	external bus voltages
I_{qi}, I_{di}	generator currents
T'_{qi}, T'_{di}	open-loop transient time-scales
T''_{qi}, T''_{di}	open-loop subtransient time-scales

TABLE I: Model parameters and variables.

Assumption 1: When using model (2), we make the following simplifying assumptions [8]:

- The frequency of each machine is operating around the synchronous frequency.
- The stator winding resistances are zero.
- The excitation voltage E_{fi} is constant for all $i \in \mathcal{V}$.
- The *subtransient saliency* is negligible, i.e. $X''_{di} = X''_{qi}, \forall i \in \mathcal{V}$.

The latter assumption is valid for synchronous machines with damper windings in both the d and q axes, which is the case for most synchronous machines [8].

Remark 1: The effects of the damper windings is explicitly governed by the last two equations of (2). Consequently, there is no asynchronous damping term appearing in the frequency dynamics of the model (2), contrary to the classical swing equations.

It is standard in the power system literature to represent the equivalent synchronous machine circuits along the dq-axes as in Figure 1, [6], [8]. Here we use the conventional phasor notation $\overline{E}''_i = \overline{E}''_{qi} + \overline{E}''_{di} = E''_{qi} + jE''_{di}$ where $\overline{E}''_{qi} := E''_{qi}, \overline{E}''_{di} := jE''_{di}, j := \sqrt{-1}$, and the phasors $\overline{I}_i, \overline{V}_i$ are defined likewise [8], [10]. Remark that internal voltages E'_q, E'_d, E''_q, E''_d as depicted in Figure 1 are not necessarily at steady state but are governed by (2), where it should be noted that, by definition, the reactances of a round rotor synchronous machine satisfy $X_{di} > X'_{di} > X''_{di} > 0, X_{qi} > X'_{qi} > X''_{qi} > 0$ for all $i \in \mathcal{V}$ [6], [8].

By Assumption 1 the stator winding resistances are negligible so that synchronous machine i can be represented by a subtransient emf behind a subtransient reactance, see Figure 2 [6], [8]. As illustrated in this figure, the internal and external voltages are related to each other by [8]

$$\overline{E}''_i = \overline{V}_i + jX''_{di}\overline{I}_i, \qquad i \in \mathcal{V}. \tag{3}$$

III. MULTI-MACHINE MODEL

Consider n synchronous machines which are interconnected by RL-transmission lines and assume that the network is operating at steady state. As the currents and voltages of each synchronous machine is expressed w.r.t. its local dq-reference frame, the network equations are written as [10]

$$\overline{I} = \operatorname{diag}(e^{-j\delta_i})\mathcal{Y}\operatorname{diag}(e^{j\delta_i})\overline{E}''. \tag{4}$$

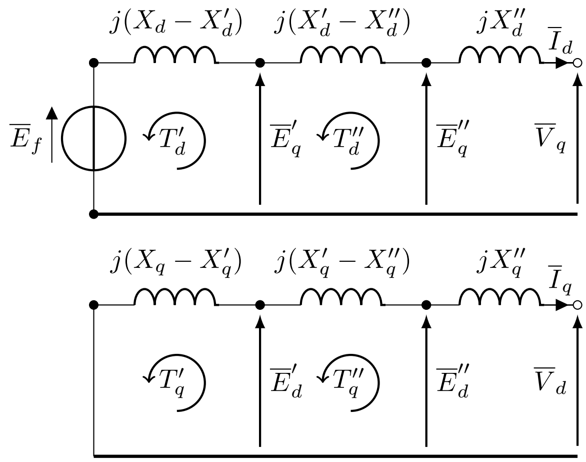

Fig. 1: Generator equivalent circuits for both dq-axes [8]. For aesthetic reasons the subscript i is dropped.

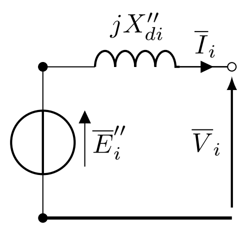

Fig. 2: Subtransient emf behind a subtransient reactance.

Here the admittance matrix[1] $\mathcal{Y} := D(R+jX)^{-1}D^T$ satisfies $\mathcal{Y}_{ik} = -G_{ik} - jB_{ik}$ and $\mathcal{Y}_{ii} = G_{ii} + jB_{ii} = \sum_{k\in\mathcal{N}_i} G_{ik} + j\sum_{k\in\mathcal{N}_i} B_{ik}$ where G denotes the conductance and $B \in \mathbb{R}^{n\times n}_{\leq 0}$ denotes the susceptance of the network [10]. In addition, $\mathcal{N}_i$ denotes the set of neighbors of node i.

Remark 2: As the electrical circuit depicted in Figure 2 is in steady state (3), the reactance X''_{di} can also be considered as part of the network (an additional inductive line) and is therefore implicitly included into the network admittance matrix $\mathcal{Y}$, see also Figure 3.

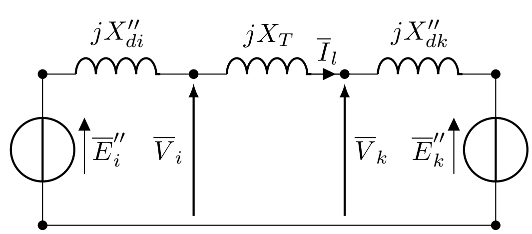

Fig. 3: Interconnection of two synchronous machines by a purely inductive transmission line with reactance X_T.

To simplify the analysis further, we assume that the network resistances are negligible so that $G = 0$. By equating the real and imaginary part of (4) we obtain the following expressions for the dq-currents entering generator $i \in \mathcal{V}$:

$$\begin{aligned} I_{di} &= B_{ii}E''_{qi} - \sum_{k\in\mathcal{N}_i}\left[B_{ik}(E''_{dk}\sin\delta_{ik} + E''_{qk}\cos\delta_{ik})\right], \\ I_{qi} &= -B_{ii}E''_{di} - \sum_{k\in\mathcal{N}_i}\left[B_{ik}(E''_{qk}\sin\delta_{ik} - E''_{dk}\cos\delta_{ik})\right], \end{aligned} \tag{5}$$

where $\delta_{ik} := \delta_i - \delta_k$. By substituting (5) and (3) into (2) we obtain after some rewriting a sixth-order multi-machine model given by equation (6), illustrated at the top of the next page.

Remark 3: Since the transmission lines are purely inductive by assumption, there are no energy losses in the transmission lines implying that the following energy conservation law holds: $\sum_{i\in\mathcal{V}} P_{ei} = 0$ where $P_{ei} = \mathrm{Re}(\overline{E}_i\overline{I}_i^*) = E''_{di}I_{di} + E''_{qi}I_{qi}$ is the electrical power produced by synchronous machine i.

IV. ENERGY FUNCTIONS

When analyzing the stability of the multi-machine system one often searches for a suitable Lyapunov function. A natural starting point is to consider the physical energy as a candidate Lyapunov function. Moreover, when we have an expression for the energy, a port-Hamiltonian representation of the associated multi-machine model (6) can be derived, see Section V.

Remark 4: It is convenient in the definition of the Hamiltonian to multiply the energy stored in the synchronous machine and the transmission lines by the synchronous frequency ω_s since a factor ω_s^{-1} appears in each of the energy functions. As a result, the Hamiltonian has the dimension of power instead of energy. Nevertheless, we still refer to the Hamiltonian as the energy function in the sequel.

In the remainder of this section we will first identify the electrical and mechanical energy stored in each synchronous machine. Next, we identify the energy stored in the transmission lines.

A. Synchronous Machine

1) Electrical Energy: Note that, at steady state, the energy (see Remark 4) stored in the first two reactances[2] of generator i as illustrated in Figure 1 is given by

$$\begin{aligned} H_{edi} &= \frac{1}{2}\left(\frac{(E'_{qi} - E_{fi})^2}{X_{di} - X'_{di}} + \frac{(E'_{qi} - E''_{qi})^2}{X'_{di} - X''_{di}}\right) \\ H_{eqi} &= \frac{1}{2}\left(\frac{(E'_{di})^2}{X_{qi} - X'_{qi}} + \frac{(E'_{di} - E''_{di})^2}{X'_{qi} - X''_{qi}}\right). \end{aligned} \tag{7}$$

Remark 5: The energy stored in the third (subtransient) reactance will be considered as part of the energy stored in the transmission lines, see also Remark 2 and Section IV-B.

[1]Recall that D is the incidence matrix of the network defined by (1).

[2]In both the d- and the q-axes.

$$
\begin{aligned}
M_i\Delta\dot{\omega}_i &= P_{mi} - P_{di} + \sum_{k\in\mathcal{N}_i} B_{ik}\Big[(E''_{di}E''_{dk} + E''_{qi}E''_{qk})\sin\delta_{ik} + (E''_{di}E''_{qk} - E''_{qi}E''_{dk})\cos\delta_{ik}\Big] \\
\dot{\delta}_i &= \Delta\omega_i \\
T'_{di}\dot{E}'_{qi} &= E_{fi} - E'_{qi} + (X_{di} - X'_{di})(B_{ii}E''_{qi} - \sum_{k\in\mathcal{N}_i}\big[B_{ik}(E''_{dk}\sin\delta_{ik} + E''_{qk}\cos\delta_{ik})\big]) \\
T'_{qi}\dot{E}'_{di} &= -E'_{di} + (X_{qi} - X'_{qi})(B_{ii}E''_{di} - \sum_{k\in\mathcal{N}_i}\big[B_{ik}(E''_{dk}\cos\delta_{ik} - E''_{qk}\sin\delta_{ik})\big]) \\
T''_{di}\dot{E}''_{qi} &= E'_{qi} - E''_{qi} + (X'_{di} - X''_{di})(B_{ii}E''_{qi} - \sum_{k\in\mathcal{N}_i}\big[B_{ik}(E''_{dk}\sin\delta_{ik} + E''_{qk}\cos\delta_{ik})\big]) \\
T''_{qi}\dot{E}''_{di} &= E'_{di} - E''_{di} + (X'_{qi} - X''_{qi})(B_{ii}E''_{di} - \sum_{k\in\mathcal{N}_i}\big[B_{ik}(E''_{dk}\cos\delta_{ik} - E''_{qk}\sin\delta_{ik})\big])
\end{aligned}
\tag{6}
$$

2) Mechanical Energy: The kinetic energy of synchronous machine i is given by

$$H_{mi} = \frac{1}{2}M_i\omega_i^2 = \frac{1}{2}M_i^{-1}p_i^2,$$

where $p_i = M_i\omega_i$ is the angular momentum of synchronous machine i with respect to the synchronous rotating reference frame.

B. Inductive Transmission Lines

Consider an interconnection between two synchronous machines with a purely inductive transmission line (with reactance X_T) at steady state, see Figure 3. When expressed in the local dq-reference frame of generator i, we observe from Figure 3 that at steady state one obtains[3]

$$jX_l\overline{I}_l = \overline{E}''_i - e^{-j\delta_{ik}}\overline{E}''_k, \tag{8}$$

where the total reactance between the internal buses of generator i and k is given by $X_l := X''_{di} + X_T + X''_{dk}$. Note that at steady state the modified energy of the inductive transmission line l between nodes i and k is given by $H_l = \frac{1}{2}X_l\overline{I}_l^*\overline{I}_l$, which by (8) can be rewritten as

$$
\begin{aligned}
H_l = -\frac{1}{2}B_{ik}\Big(&2\left(E''_{di}E''_{qk} - E''_{dk}E''_{qi}\right)\sin\delta_{ik} \\
&-2\left(E''_{di}E''_{dk} + E''_{qi}E''_{qk}\right)\cos\delta_{ik} \\
&+E''^2_{di} + E''^2_{dk} + E''^2_{qi} + E''^2_{qk}\Big),
\end{aligned}
\tag{9}
$$

where the line susceptance satisfies $B_{ik} = -\frac{1}{X_l} < 0$ [10].

C. Total Energy

The total physical energy of the multi-machine system is equal to the sum of the individual energy functions:

$$H = \sum_{i\in\mathcal{V}}(H_{edi} + H_{eqi} + H_{mi}) + \sum_{l\in\mathcal{E}} H_l. \tag{10}$$

V. PORT-HAMILTONIAN REPRESENTATION

Using the energy functions from the previous section, the multi-machine model (6) can be put into a port-Hamiltonian form. To this end, we derive expressions for the gradient of each energy function.

[3]The mapping from dq-reference frame k to dq-reference frame i in the phasor domain is done by multiplication of $e^{-j\delta_{ik}}$ [10].

A. Transmission Line Energy

Recall that the energy stored in transmission line l between internal buses i and k is given by (9). It can be verified that the gradient of the *total* energy $H_L := \sum_{l\in\mathcal{E}} H_l$ stored in the transmission lines takes the form

$$\begin{bmatrix}\frac{\partial H_L}{\partial \delta_i}\\ \frac{\partial H_L}{\partial E''_{qi}}\\ \frac{\partial H_L}{\partial E''_{di}}\end{bmatrix} = \begin{bmatrix}E''_{di}I_{di} + E''_{qi}I_{qi}\\ -I_{di}\\ I_{qi}\end{bmatrix} = \begin{bmatrix}P_{ei}\\ -I_{di}\\ I_{qi}\end{bmatrix},$$

where I_{di}, I_{qi} are given by (5). Here it is used that the self-susceptances satisfy $B_{ii} = \sum_{k\in\mathcal{N}_i} B_{ik}$ for all $i\in\mathcal{V}$.

1) State transformation: In the sequel, it is more convenient to consider a different set of variable describing the voltage angle differences. Define for each edge $l\in\mathcal{E}$ $\eta_l := \delta_{ik}$ where i, k are respectively the positive and negative ends of l. In vector form we obtain $\eta = D^T\delta \in \mathbb{R}^m$, and observe that this implies $D\frac{\partial H}{\partial \eta} = D\frac{\partial H_L}{\partial \eta} = P_e$.

B. Electrical Energy of the Synchronous Generator

Further, notice that the electrical energy stored in the equivalent circuits along the d- and q-axis of generator i is given by (7) and satisfies

$$\begin{bmatrix}X_{di} - X'_{di} & X_{di} - X'_{di}\\ 0 & X'_{di} - X''_{di}\end{bmatrix}\begin{bmatrix}\frac{\partial H_{edi}}{\partial E'_{qi}}\\ \frac{\partial H_{edi}}{\partial E''_{qi}}\end{bmatrix} = \begin{bmatrix}E'_{qi} - E_{fi}\\ E''_{qi} - E'_{qi}\end{bmatrix}$$

$$\begin{bmatrix}X_{qi} - X'_{qi} & X_{qi} - X'_{qi}\\ 0 & X'_{qi} - X''_{qi}\end{bmatrix}\begin{bmatrix}\frac{\partial H_{eqi}}{\partial E'_{di}}\\ \frac{\partial H_{eqi}}{\partial E''_{di}}\end{bmatrix} = \begin{bmatrix}E'_{di}\\ E''_{di} - E'_{di}\end{bmatrix}.$$

By the previous observations, and by aggregating the states, the dynamics of the multi-machine system can now be written in the form (11), see next page, where the Hamiltonian is given by (10) and $\hat{X}_{di} := X_{di} - X'_{di}, \hat{X}'_{di} := X'_{di} - X''_{di}, \hat{X}_d = \mathrm{diag}_{i\in\mathcal{V}}\{\hat{X}_{di}\}$ and $\hat{X}'_d, \hat{X}_q, \hat{X}'_q$ are defined likewise. In addition, $T'_d = \mathrm{diag}_{i\in\mathcal{V}}\{T'_{di}\}$ and T'_d, T_q, T'_q are defined similarly. Observe that the multi-machine system (11) is of the form

$$
\begin{aligned}
\dot{x} &= (J - R)\nabla H(x) + gu \\
y &= g^T\nabla H(x)
\end{aligned}
\tag{12}
$$

where $J = -J^T$, $R = R^T$ are respectively the anti-symmetric and symmetric part of the matrix depicted in (11).

$$\dot{x}=\begin{bmatrix}\dot{p}\\ \dot{\eta}\\ \dot{E}'_q\\ \dot{E}'_d\\ \dot{E}''_q\\ \dot{E}''_d\end{bmatrix}=\begin{bmatrix}0 & -D & 0 & 0 & 0 & 0\\ D^T & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -(T'_d)^{-1}\hat{X}_d & 0 & -(T'_d)^{-1}\hat{X}_d & 0\\ 0 & 0 & 0 & -(T'_q)^{-1}\hat{X}_q & 0 & -(T'_q)^{-1}\hat{X}_q\\ 0 & 0 & 0 & 0 & -(T''_d)^{-1}\hat{X}'_d & 0\\ 0 & 0 & 0 & 0 & 0 & -(T''_q)^{-1}\hat{X}'_q\end{bmatrix}\nabla H+g(P_m-P_d), \qquad (11)$$
$$y=g^T\nabla H=M^{-1}p=\omega, \qquad g=\begin{bmatrix}I & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T.$$

Notice that the dissipation matrix of the *electrical part* is positive definite (which implies $R\geq 0$) if

$$\begin{bmatrix}2\frac{X_{di}-X'_{di}}{T'_{di}} & \frac{X_{di}-X'_{di}}{T'_{di}}\\ \frac{X_{di}-X'_{di}}{T'_{di}} & 2\frac{X'_{di}-X''_{di}}{T''_{di}}\end{bmatrix}>0, \qquad \forall i\in\mathcal{V},$$

which, by invoking the Schur complement, holds if and only if

$$4(X'_{di}-X''_{di})T'_{di}-(X_{di}-X'_{di})T''_{di}>0, \quad \forall i\in\mathcal{V}. \quad (13)$$

Note that a similar condition holds for the q-axis.

Proposition 1: Suppose that for all $i\in\mathcal{V}$ the following holds:

$$\begin{aligned}4(X'_{di}-X''_{di})T'_{di}-(X_{di}-X'_{di})T''_{di}>0\\ 4(X'_{qi}-X''_{qi})T'_{qi}-(X_{qi}-X'_{qi})T''_{qi}>0.\end{aligned} \quad (14)$$

Then (11) is a port-Hamiltonian representation of the multi-machine network (6).

It should be stressed that (14) is not a restrictive assumption since $T''_{di}\ll T'_{di}, T''_{qi}\ll T'_{qi}$ for a typical generator, see also Table 4.2 of [6] and Table 4.3 of [8].

Because the interconnection and damping structure $J-R$ of (11) is state-independent, the shifted Hamiltonian

$$\bar{H}(x)=H(x)-(x-\bar{x})^T\nabla H(\bar{x})-H(\bar{x}) \quad (15)$$

acts as a local storage function for proving passivity in a neighborhood of a steady state $\bar{x}$ of (12), provided that the Hessian of H evaluated at $\bar{x}$ (denoted as $\nabla^2 H(\bar{x})$) is positive definite[4].

Proposition 2: Let $\bar{u}$ be a constant input and suppose there exists a corresponding steady state $\bar{x}$ to (12) such that $\nabla^2 H(\bar{x})>0$. Then the system (12) is passive in a neighborhood of $\bar{x}$ with respect to the shifted external port-variables $\tilde{u}:=u-\bar{u}, \tilde{y}:=y-\bar{y}$ where $\bar{y}:=g^T\nabla H(\bar{x})$.

Proof: Define the shifted Hamiltonian by (15), then we obtain

$$\begin{aligned}\dot{x}&=(J-R)\nabla H(x)+gu\\ &=(J-R)(\nabla\bar{H}(x)+\nabla H(\bar{x}))+gu\\ &=(J-R)\nabla\bar{H}(x)+g(u-\bar{u})\\ &=(J-R)\nabla\bar{H}(x)+g\tilde{u}\\ \tilde{y}&=y-\bar{y}=g^T(\nabla H(x)-\nabla H(\bar{x}))=g^T\nabla\bar{H}(x).\end{aligned} \quad (16)$$

As $\nabla^2 H(\bar{x})>0$ we have that $\bar{H}(\bar{x})=0$ and $\bar{H}(x)>0$ for all $x\neq\bar{x}$ in a sufficiently small neighborhood around $\bar{x}$. Hence, by (16) the passivity property automatically follows where $\bar{H}$ acts as a local storage function. ■

[4]Observe that $\nabla^2 H(x)=\nabla^2\bar{H}(x)$ for all x.

VI. MINIMIZING GENERATION COSTS

The objective is to minimize the total quadratic generation cost while achieving zero frequency deviation. By analyzing the steady states of (6), it follows that a necessary condition for zero frequency deviation is $\mathbb{1}^T P_m=\mathbb{1}^T P_d$, i.e., the total supply must match the total demand. Therefore, consider the following convex minimization problem:

$$\begin{aligned}\min_{P_m}\ &\frac{1}{2}P_m^T Q P_m\\ \text{s.t. }&\mathbb{1}^T P_m=\mathbb{1}^T P_d,\end{aligned} \quad (17)$$

where $Q=Q^T>0$ and P_d is a constant *unknown* power load.

Remark 6: Note that that minimization problem (17) is easily extended to quadratic cost functions of the form $\frac{1}{2}P_m^T Q P_m+b^T P_m$ for some $b\in\mathbb{R}^n$. Due to space limitations, this extension is omitted.

As the minimization problem (17) is convex, it follows that P_m is an optimal solution if and only if the Karush-Kuhn-Tucker conditions are satisfied [3]. Hence, the optimal points of (17) are characterized by

$$P_m^*=Q^{-1}\mathbb{1}\lambda^*, \qquad \lambda^*=\frac{\mathbb{1}^T P_d}{\mathbb{1}^T Q^{-1}\mathbb{1}} \quad (18)$$

Next, based on the design of [16], consider a distributed controller of the form

$$\begin{aligned}T\dot{\theta}&=-L_c\theta-Q^{-1}\omega\\ P_m&=Q^{-1}\theta-K\omega\end{aligned} \quad (19)$$

where $T=\text{diag}_{i\in\mathcal{V}}\{T_i\}>0, K=\text{diag}_{i\in\mathcal{V}}\{k_i\}>0$ are controller parameters and $\theta\in\mathbb{R}^n$ is the controller variable. In addition, L_c is the Laplacian matrix of some connected undirected weighted communication graph.

The controller (19) consists of three parts. Firstly, the term $-K\omega$ corresponds to a primary controller and adds damping into the system. The term $-Q^{-1}\omega$ corresponds to secondary control for guaranteeing zero frequency deviation on longer time-scales. Finally, the term $-L_c\theta$ corresponds to tertiary control for achieving optimal production allocation over the network.

Note that (19) admits the port-Hamiltonian representation

$$\begin{aligned}\dot{\vartheta}&=-L_c\nabla H_c-Q^{-1}\omega\\ P_m&=Q^{-1}\nabla H_c-K\omega, \qquad H_c=\frac{1}{2}\vartheta^T T^{-1}\vartheta,\end{aligned} \quad (20)$$

where $\vartheta := T\theta$. By interconnecting the controller (20) with (11), the closed-loop system amounts to

$$\begin{bmatrix}\dot{x}\\ \dot{\vartheta}\end{bmatrix} = \begin{bmatrix}J-R-R_K & G^T\\ -G & -L_c\end{bmatrix}\nabla H_e - \begin{bmatrix}g\\ 0\end{bmatrix}P_d \qquad (21)$$
$$G = \begin{bmatrix}Q^{-1} & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

where $J-R$ is given as in (11), $H_e := H + H_c$, and $R_K = \mathrm{blockdiag}(0, K, 0, 0, 0, 0)$. Define the set of steady states of (21) by Ω and observe that any $x_e := (x, \vartheta) \in \Omega$ satisfies the optimality conditions (18) and $\omega = 0$.

Assumption 2: $\Omega \neq \emptyset$ and there exists $\bar{x}_e \in \Omega$ such that $\nabla^2 H_e(\bar{x}_e) > 0$.

Remark 7: While the Hessian condition of Assumption 2 is required for proving local asymptotic stability of (21), guaranteeing that this condition holds can be bothersome. However, while we omit the details, it can be shown that $\nabla^2 H_e(\bar{x}_e) > 0$ if

- the generator reactances are small compared to the transmission line reactances.
- the subtransient voltage differences are small.
- the rotor angle differences are small.

Theorem 1: Suppose P_d is constant and there exists $\bar{x}_e \in \Omega$ such that Assumption 2 is satisfied. Then the trajectories of the closed-loop system (21) initialized in a sufficiently small neighborhood around $\bar{x}_e$ converge to the set of optimal points Ω.

Proof: Observe by (16) that the shifted Hamiltonian defined by (15) satisfies

$$\dot{\bar{H}}_e = -(\nabla \bar{H}_e)^T \,\mathrm{blockdiag}(R + R_K, L_c)\nabla \bar{H}_e \leq 0$$

where equality holds if and only if $\omega = 0$, $T^{-1}\vartheta = \theta = \mathbb{1}\theta^*$ for some $\theta^* \in \mathbb{R}$, and $\nabla_E \bar{H}_e(x_e) = \nabla_E H_e(x_e) = 0$. Here $\nabla_E H_e(x_e)$ is the gradient of H_e with respect to the internal voltages E_q', E_d', E_q'', E_d''. By Assumption 2 there exists a compact neighborhood Υ around $\bar{x}_e$ which is forward invariant. By invoking LaSalle's invariance principle, trajectories initialized in Υ converge to the largest invariant set where $\dot{\bar{H}}_e = 0$. On this set $\omega, \eta, \theta, E_q', E_d', E_q'', E_d''$ are constant and, more specifically, $\omega = 0$, $\theta = \mathbb{1}\lambda^* = \mathbb{1}\frac{\mathbb{1}^T P_d}{\mathbb{1}^T Q^{-1}\mathbb{1}}$ corresponds to an optimal point of (17) as $P_m = Q^{-1}\mathbb{1}\lambda^*$ where λ^* is defined in (18). We conclude that the trajectories of the closed-loop system (21) initialized in a sufficiently small neighborhood around $\bar{x}_e$ converge to the set of optimal points Ω. ■

VII. CONCLUSIONS

We have shown that a much more advanced multi-machine model than conventionally used can be analyzed using the port-Hamiltonian framework. Based on the energy functions of the system, a port-Hamiltonian representation of the model is obtained. Moreover, the system is proven to be incrementally passive which allows the use of a passive controller that regulates the frequency in an optimal manner, even in the presence of an unknown constant demand.

The results established in this paper can be extended in many possible ways. Current research has shown that the third, fourth and fifth order model as given in [8] admit a similar port-Hamiltonian structure as (11). It is expected that the same controller as designed in this paper (or other passive optimal controllers as considered in our previous work [13], [14], [15]) can also be used in these lower order models.

While the focus in this paper is about (optimal) frequency regulation, further effort is required to investigate the possibilities of (optimal) voltage control using passive controllers. Another extension is to consider the case where inverters and frequency dependent loads are included into the network as well. Finally, one could look at the possibility to include transmission line resistances in the network.

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